Journal of Sound and Vibration (1997) **202**(4), 593–599

GENERALIZED POWER SERIES SOLUTIONS OF THE VIBRATION OF CLASSICAL CIRCULAR PLATES WITH VARIABLE THICKNESS

J. WANG

Epson Palo Alto Laboratory, 3145 *Porter Drive*, *Suite* 104, *Palo Alto*, *CA* 94304-1224, *U.S.A.*

(*Received* 17 *January* 1995, *and in final form* 3 *September* 1996)

1. INTRODUCTION

The vibration of classical circular plates of variable thickness has been studied by many investigators for a very long period. All the results have been thoroughly summarized by Leissa in his landmark reference [1] and consequent update [2].

Analytical solutions for axisymmetric vibrations in terms of Bessel functions for particular Poisson ratios were given by Conway [3, 4] for the thickness variation described by a power function and by Conway *et al*. [5] for the thickness variation described in a linear manner. By employing the Forbenius method, the vibration equations of axisymmetric vibrations were solved by Jain *et al*. [6] for the linear case, the solutions being given in infinite power series. Using the perturbation method, Yang [7] also studied the linear thickness variation case.

In the present paper, the vibrations of plates with generalized variable thickness are studied, and the power series solutions are given. These solutions, represented by the recursive relations of the coefficients of the infinite power series, can be applied to various boundary conditions to obtain the resonance frequency spectra and mode shapes. These infinite power series, which are Bessel functions for a particular Poisson ratio and other parameters, can be effectively evaluated without special difficulty for most parameters involved.

2. AXISYMMETRIC VIBRATION EQUATIONS OF CIRCULAR PLATES

The axisymmetric vibration equation of a circular isotropic plate [1] is

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left[D\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+\frac{\partial D}{\partial r}\left(\frac{\partial^2 w}{\partial r^2}+\frac{v}{r}\frac{\partial w}{\partial r}\right)\right]\right\}-\omega^2\rho h w=0, \tag{1}$$

where w is the transverse displacement, v is the Poisson ratio, ρ is the density of the material, ω is the frequency, and $h=h(r)$ is the thickness of the plate. The stiffness of the plate is

$$D=Eh^3/12(1-v^2), \tag{2}$$

where E is Young's modulus of the material. The moments and shear force are

$$M_r=-D\left[\frac{\partial^2 w}{\partial r^2}+\frac{v}{r}\frac{\partial w}{\partial r}\right],\qquad M_\theta=-D\left[v\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right],\qquad Q_r=\frac{1}{r}\left[\frac{\partial}{\partial r}(rM_r)-M_\theta\right]. \tag{3}$$

0022–460X/97/190593 + 07 $25.00/0/sv960810

Now one expands equation (1), and by taking into account that the thickness h is a function of r, one has

$$H^2R^3W'''' + 2HR^2(H + 3H'R)W''' + R[3R^2(2H'^2 + HH'') + 3(2 + \nu)HH'R - H^2]W''$$
$$+ [3\nu R^2(2H'^2 + HH'') - 3HH'R + H^2]W' - \Omega^2 R^3 W = 0, \quad (4)$$

where the dimensionless quantities are

$$R = \frac{r}{a}, \qquad W = w(R), \qquad H = \frac{h(R)}{h_0}, \qquad \Omega = \frac{\omega}{\omega_0}, \qquad \omega_0 = \frac{h_0}{a^2}\sqrt{\frac{E}{12(1 - \nu^2)\rho}}, \quad (5)$$

where a is the radius of the plate and h_0 is the thickness at the center of the plate.

Now one can see that the equation is being determined by the dimensionless thickness function H. For some special H, there may be analytical solutions, like the uniform case, which can be represented by letting $H = 1$. In this study, it will be assumed that the function is a power function, and two special cases, $H = 1 + \xi R^m$ and $H = R^m$, will be solved in infinite power series.

3. POWER SERIES SOLUTIONS FOR $H = 1 + \xi R^m$

It is clear now that the differential equation of motion is of fourth order with variable coefficients. Generally there is no closed form solution in tabulated functions except in some special cases [3–5], which are Bessel functions. Since Bessel functions themselves are obtained from second order differential equations by the Forbenius method, one can employ this method for the above equation, and be able to get a group of power series solutions which are generalized functions, with Bessel functions as a special case.

Here the thickness variation parameter is defined as

$$\xi = h_1/h_0 - 1, \quad (6)$$

where h_1 is the thickness of the plate at the edge, where $R = 1$ or $r = a$, and h_0 is the thickness at the center, where $R = 0$ or $r = 0$. When $\xi = 0$, one has a uniform plate. By substituting for H in equation (4), one has

$$(1 + \xi R^m)^2 R^3 W'''' + 2(1 + \xi R^m)R^2[1 + \xi(1 + 3m)R^m]W'''$$
$$+ R\{[3m(1 + \nu + 3m) - 1]\xi^2 R^{2m} + [3m(1 + \nu + m) - 2]\xi R^m - 1\}W''$$
$$+ \{[3m(3m\nu - \nu - 1) + 1]\xi^2 R^{2m} + [3m(m\nu - \nu - 1) + 2]\xi R^m + 1\}W' - \Omega^2 R^3 W = 0. \quad (7)$$

Let the solution be

$$W = \sum_{n=0}^{\infty} a_n R^{n+s}, \qquad a_0 \neq 0. \quad (8)$$

A substitution of equation (8) into equation (7) yields

$$\sum_{n=0}^{\infty} \{ f_1(n)a_n R^{n+s-1} + f_2(n)a_n R^{n+s-1+m} + f_3(n)a_n R^{n+s-1+2m} - \Omega^2 a_n R^{n+s+3}\} = 0, \quad (9)$$

with

$$f_1(n) = (n+s)^2(n+s-2)^2,$$
$$f_2(n) = \xi(n+s)\{2(n+s-1)(n+s-2)(n+s+3m-1) + [3m(m+v+1)-2](n+s-1) + 3m(mv-v+1)+2\},$$
$$f_3(n) = \xi^2\{(n+s)^2(n+s-2)^2 + 3m(n+s)(n+s-1)[2(n+s)+3(m-1)+v] + 3m(n+s)(3mv-v-1)\}. \qquad (10)$$

From equation (9), one obtains the indicial equation as

$$s^2(s-2)^2 = 0, \qquad \text{or} \qquad s = 0, 0, 2, 2. \qquad (11)$$

Also it is clear that, for $m = 1, 2$, a degree of four recursive relation will be obtained from equation (9). For $m \geqslant 3$, the degree of the recursive relation will be $2m$.

Now one investigates the case $m = 1$ in detail. From equation (9) one has the recursive relation for the coefficients of equation (8) as

$$a_{n+4} = -[(n+s+3)\xi/(n+s+2)^2(n+s+4)^2] \times\{(n+s+2)[2(n+s+1)(n+s+5)+4+3v]-1\}a_{n+3} - [\xi^2/(n+s+2)(n+s+4)^2] \times\{(n+s+1)[(n+s)(n+s+7)+11+3v]+2(3v-1)\}a_{n+2} + [\Omega^2/(n+s+2)^2(n+s+4)^2]a_n, \qquad (12)$$

for $n = 0, 1, 2, \ldots, \infty$. With $s = 0$, one finds that there are two independent coefficients obtained from equations (9) and (12), and thus two independent solutions. For $s = 2$, one obtains two solutions which are identical to those from $s = 0$, so one can conclude that there are only two solutions associated with s. The other two solutions must be obtained by the Forbenius method, and are singular at $R = 0$. Also, from the above equation one can see that the series will be convergent for $R < |1/\xi|$.

By assuming

$$a_0 = A + B, \qquad a_2 = A - B, \qquad (13)$$

where A and B are two arbitrary constants, one obtains

$$a_n = f_1(n)A + f_2(n)B, \qquad (14)$$

and the first few terms of $f_i(n)$ for $i = 1, 2$ are

$$f_1(0) = 1, \quad f_1(1) = 0, \quad f_1(2) = 1, \quad f_1(3) = -2\xi(1+v)/3,$$
$$f_1(4) = (\Omega/8)^2 + \tfrac{3}{32}(1+v)(3+2v)\xi^2 \ldots,$$
$$f_2(0) = 1, \quad f_2(1) = 0, \quad f_2(2) = -1, \quad f_2(3) = 2\xi(1+v)/3,$$
$$f_2(4) = (\Omega/8)^2 - \tfrac{3}{32}(1+v)(3+2v)\xi^2 \ldots, \qquad (15)$$

and the two solutions are

$$W_i(R, v, \xi) = \sum_{n=0}^{\infty} f_i(n)R^n, \qquad i = 1, 2. \qquad (16)$$

To obtain the singular solutions at $R = 0$, one can assume that the solutions are

$$W_3(R, v, \xi) = C_1 W_1(R, v, \xi) \ln R + \sum_{n=0}^{\infty} b_n R^n, \qquad b_0 \neq 0;$$

$$W_4(R, v, \xi) = C_2 W_2(R, v, \xi) \ln R + \sum_{n=0}^{\infty} c_n R^n, \qquad c_0 \neq 0, \tag{17}$$

and substitute them into equation (7) with known W_1 and W_2 from equation (16); coefficients b_n and c_n and constants C_1 and C_2 can be determined by another two recursive relations, thus a pair of solutions in the form of power series will be obtained. With these two, the solutions set for equation (7) is now complete, and the annular plates can also be studied.

For cases with $m \geqslant 1$, one can obtain the solutions in a similar manner.

As a numerical example, the resonance frequencies for various ξ are compared in Table 1 with the approximate results from perturbation and FEM by Yang [7]. From the above computations, it is found that the power series functions converge very quickly, and the computer programming is also straightforward. There is no difficulty in obtaining these results. From the result at $\xi = 0$ one can see that the exact frequency is obtained for the uniform thickness plate from the power series solution.

4. POWER SERIES SOLUTIONS FOR $H = R^m$

In this case, the thickness of the plate at the center, which is $R = 0$, is zero, and h_0 will be the thickness at the edge.

From equation (4), the equation of motion in this case becomes

$$R^{2m} W'''' + 2(1 + 3m) R^{2m-1} W''' + [3m(1 + 3m + v) - 1] R^{2(m-1)} W'' + \{3m[3vm - (1 + v)] + 1\} R^{2m-3} W' - \Omega^2 W = 0. \tag{18}$$

TABLE 1

Comparison of exact Ω^2 for $v = 0{\cdot}3$ with approximate results by Yang [7] for clamped circular plates

ξ	Perturbation [7]	FEM [7]	Present
−0·70	–	–	15·7528
−0·50	–	–	42·2463
−0·30	–	–	67·8149
−0·20	–	–	80·2763
−0·10	86·8	92·9	92·4751
−0·05	95·1	99·1	98·4612
0·00	103·4	105·3	104·3631
0·05	111·7	111·6	110·1741
0·10	120·0	118·2	115·8873
0·20	136·7	131·6	126·9920
0·30	153·3	145·8	137·6199
0·50	–	–	157·2105
0·70	–	–	174·1848

Now let one examine (18) for different m. For $m = 1$, by substituting equation (8) into equation (18), one has

$$\sum_{n=0}^{\infty} \{ f_1(n) a_n R^{n+s-1} - \Omega^2 a_n R^{n+s+1} \} = 0, \tag{19}$$

where

$$f_1(n) = (n+s)[(n+s-1)(n+s-2)(n+s+5) + (11+3v)(n+s-1) + 2(3v-1)], \tag{20}$$

and the indicial equation will be

$$s[(s-1)(s-2)(s+5) + (11+3v)(s-1) + 2(3v-1)] = 0, \tag{21}$$

which will give

$$s = -1,\ -\tfrac{1}{2} - \sqrt{13-12v}/2,\ 0,\ -\tfrac{1}{2} + \sqrt{13-12v}/2. \tag{22}$$

The coefficients of the series will be again given by a recursive relation,

$$a_{n+2} = \frac{\Omega^2 a_n}{(n+s+2)\{(n+s+1)[(n+s)(n+s+7) + 11 + 3v] + 2(3v-1)\}}. \tag{23}$$

The infinite series will be of the R^2 type.

Since $-1 < v < \frac{1}{2}$, one has four distinct solutions for each s, and two of the solutions are finite at $R = 0$, which correspond to the two larger roots.

For $m = 2$, equation (18) will be an Euler equation, and by using $z = \ln R$, can be transformed into

$$W'''' + 8W''' + 2(5+3v)W'' - 24(1-v)W' - \Omega^2 W = 0. \tag{24}$$

Assuming $W(z) = e^{\lambda z} = R^{\lambda}$, the characteristic equation will be

$$\lambda^4 + 8\lambda^3 + 2(5+3v)\lambda^2 - 24(1-v)\lambda - \Omega^2 = 0, \tag{25}$$

and the solutions are

$$\lambda_{1,2} = -2 \pm \sqrt{7 - 3v + \sqrt{9(1-v)^2 + \Omega^2}}, \qquad \lambda_{1,2} = -2 \pm \sqrt{7 - 3v - \sqrt{9(1-v)^2 + \Omega^2}}. \tag{26}$$

This has been studied by Conway [3]. Due to the transformation $z = \ln R$, these solutions may be singular at $R = 0$, and therefore these solutions cannot be applied to the vibration of a circular plate.

For $m = 3$, by applying a transformation $z = 1/R$, one has a new equation

$$z^3 W'''' - 8z^2 W''' + (5+9v) z W'' + 18(5-3v)W' - \Omega^2 z W = 0. \tag{27}$$

Again, the four solutions of the indicial equation are

$$s = 7 - \sqrt{85-36v}/2,\ 0,\ 7,\ 7 + \sqrt{85-36v}/2, \tag{28}$$

and the recursive equation for the coefficients is

$$a_{n+2} = \frac{\Omega^2 a_n}{(n+s+2)\{(n+s+1)[(n+s)(n+s-9) + 5 + 9v] + 18(5-3v)\}}, \tag{29}$$

which will give solutions in terms of an infinite power series of $(1/R)^2$. The interesting phenomenon is that these solutions are singular at $R = 0$ due to the transformation

TABLE 2

First three frequencies Ω of a clamped plate with $H = R$ for various Poisson ratios

$\nu = 1/3$ Conway [4]	$\nu = 1/3$	$\nu = 0{\cdot}30$	$\nu = 0{\cdot}25$
8·72	8·7193	8·8194	8·9656
21·15	21·1457	21·3340	21·6094
38·45	38·4538	38·7253	39·1222

$z = 1/R$, which means these solutions cannot be applied to the cases in which the plate is a circular plate rather than an annular one. Apparently, it is conceivable that for $m \geqslant 3$, the displacements at the center of the plate may be so large that the vibration could not occur, given the fact that the stiffness of the plate close to the center is very small. With this in mind, the singular solutions at $R = 0$ will not be hard to understand, and then it can be concluded that for $m \geqslant 2$, there is no more vibration, and the natural frequency will be zero. But if the plate is an annular one, the fast convergent solutions can be applied to analyze the frequency spectra.

For $m \geqslant 3$, one needs transformations such as $z = 1/R^n$, where $n \geqslant 1$. This will enable one to achieve solutions by the Frobenius method similar to the previous ones. And, as one can see from the $m = 2$ case, one needs to be careful to distinguish the solutions which are singular at $R = 0$, or at the center of the plate.

It is also interesting to note that the indicial equation of the differential equation changes with m, which is different from the results of the previous section.

As a numerical example, the case for $m = 1$ is computed. It is found that the frequencies for $\nu = 1/3$ are precisely the results by Conway [4]. For comparison, the first few frequencies for ν are given in Table 2. It is found that the power series here converges even faster than in the previous case, and no difficulty in both speed and precision were experienced.

5. CONCLUSIONS

From the above solutions and numerical results, it is proven that the axisymmetric vibrations of circular plates can be analyzed with available exact solutions in terms of power series. These series, which converge rapidly for parameters which do not represent limiting cases, can be evaluated efficiently with currently available computing resources.

The singular solutions in many cases, which include some limiting cases, reflect the physical feature of axisymmetric vibrations. Keeping this in mind, the real implications of these solutions can be well interpreted to mean that the excessive displacement in the center makes the vibration impossible.

Although only two special cases of thickness variation are studied here, this method can be readily extended to other thickness variation schemes to cover a wide range of practical problems for precise analytical solutions.

REFERENCES

1. A. W. LEISSA 1969 *Vibration of Plates*. Washington, DC: NASA; Scientific and Technical Information Division, Office of Technology Utilization.
2. A. W. LEISSA 1977 *Shock and Vibration Digest* **9**, 1–35. Recent research in plate vibrations, 1973–1976: complicating effects.
3. H. D. CONWAY 1958 *Ingenieur-Archiv* **26**, 408–410. Some special solutions for the flexural vibration of discs of varying thickness.

4. H. D. Conway 1957 *Zeitschrift für angewandte Mathematik und Mechanik* **37**, 406–407. An analogy between the flexural vibrations of a cone and a disc of linearly varying thickness.
5. E. C. H. Becker, H. D. Conway and J. F. Dubil 1964 *Journal of Applied Mechanics* **31**, 329–331. Vibration frequencies of tapered bars and circular plates.
6. R. K. Jain, C. Prasad and S. R. Soni 1972 *Journal of Applied Mathematics and Physics* (*ZAMP*) **23**, 941–947. Axisymmetric vibrations of circular plates of linearly varying thickness.
7. J. S. Yang 1993 *Journal of Sound and Vibration* **165**, 178–184. The vibration of a circular plate with varying thickness.